

LINEAR PROGRAMMING : Some Worked Examples and Exercises for Grades 11 and 12 Learners.

Example :

A small business enterprise makes dresses and trousers. To make a dress requires $\frac{1}{2}$ hour of cutting and 20 minutes of stitching. To make a trousers requires 15 minutes of cutting and $\frac{1}{2}$ hour of stitching. The profit on a dress is R40 and on a pair of trousers R50. The business operates for a maximum of 8 hours per day. Determine how many dresses and trousers should be made to maximize profit and what the maximum profit is.

Solution :

Step 1 : To solve the above problem we would have to translate the conditions or **constraints** from a verbal to a symbolic form. We first introduce our variables.

Let x be the number of dresses and y the number of trousers.

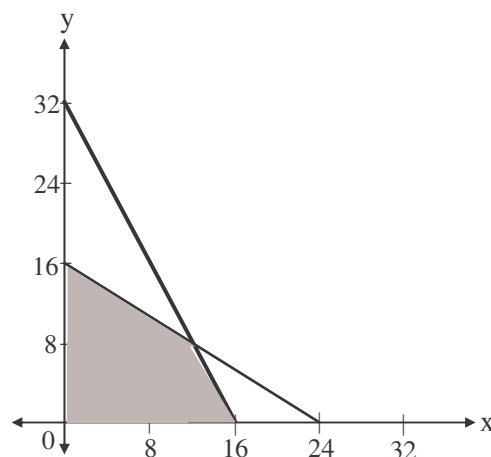
Step 2 : Next we express the constraints as a system of inequalities .

$$x \geq 0 \text{ and } y \geq 0, \text{ } x \text{ and } y \text{ being whole numbers ie. } x, y \in N_0$$

$$\text{Cutting Time: } \frac{1}{2}x + \frac{1}{4}y \leq 8 \text{ ie. } 2x + y \leq 32$$

$$\text{Stitching Time: } \frac{1}{3}x + \frac{1}{2}y \leq 8 \text{ ie. } 2x + 3y \leq 48$$

Step 3 : Graph the system of inequalities and shade in the region that satisfy the constraints. The shaded region is called the **feasible region**.



Step 4: Write the profit in terms of the variables.

$$P = 40x + 50y$$

Since the objective (in this case) is to make a maximum profit, the profit equation is called the **objective function**.

Step 5: To determine the maximum profit and the number of items that will give the maximum profit we can use one of two methods :

5.1 Substitute each of the ordered pairs of the vertices of the feasible region, excluding $(0;0)$, ie. $(0;16)$, $(16;0)$ and $(12;8)$ into profit equation .

$$\begin{array}{lll} \text{(i) } P = 40(0) + 50(16) & \text{(ii) } P = 40(16) + 50(0) & \text{(iii) } P = 40(12) + 50(8) \\ = 0 + 800 & = 640 + 0 & = 480 + 400 \\ = 800 & = 640 & = 880 \end{array}$$

The maximum profit is **R880,00** for $x = 12$ and $y = 8$.

5.2 Using the search line :

In the objective function, make y the subject of the formula:

$$\text{ie. } y = -\frac{40}{50}x + \frac{P}{50}$$

The gradient of this line is $-\frac{4}{5}$ or any multiple of this, eg. $-\frac{8}{10}$, $-\frac{12}{15}$, etc.

Draw any broken line with this gradient. For maximum profit this line, called the **SEARCH LINE**, must be moved parallel to the first line until it passes through the highest point in the feasible region. This point is $A(12;8)$. We substitute only this ordered pair in the profit equation to obtain the maximum profit as indicated above.

Steps to be followed in solving a Linear Programming Problem

1. Define the **variables** if they are not already defined in the problem, ie.
Let x be and y be
2. Write down the **constraints** in terms of the variables.
3. Graph the constraints and shade the **Feasible Region**.
4. Write down the **Objective Function** in terms of the variables.
5. Using the gradient of the objective function, draw a **Search Line**. Use this line to maximise or minimise the objective function.

Further Examples

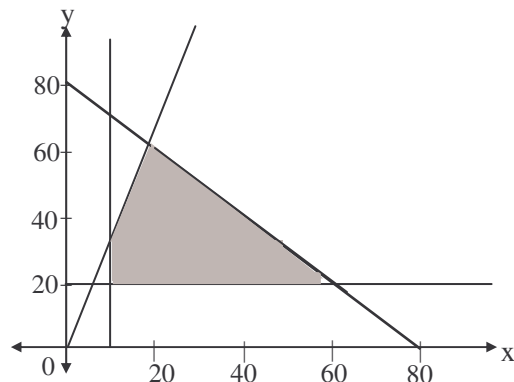
1. A farmer has 80 hectares of his farm available for planting maize and cabbages. He must grow at least 10 hectares of maize and 20 hectares of cabbages to meet demands. He prefers to plant more cabbages than maize but his work force and equipment will only allow him to cultivate a maximum of three times the quantity of cabbages to that of maize.
 - 1.1 Represent the information above as a system of inequalities.
 - 1.2 Sketch a graph of these inequalities.
 - 1.3 If the profit on maize is R800 per ha and on cabbages R500 per ha, how should the farmer plant the two crops to make a maximum profit and what is this profit.

Solution :

- 1.1 Let x be the number of hectares of maize and y the number of hectares of cabbages.

$$\begin{array}{l} x + y \leq 80 \\ x \geq 10, y \geq 20 \\ y \leq 3x \end{array}$$

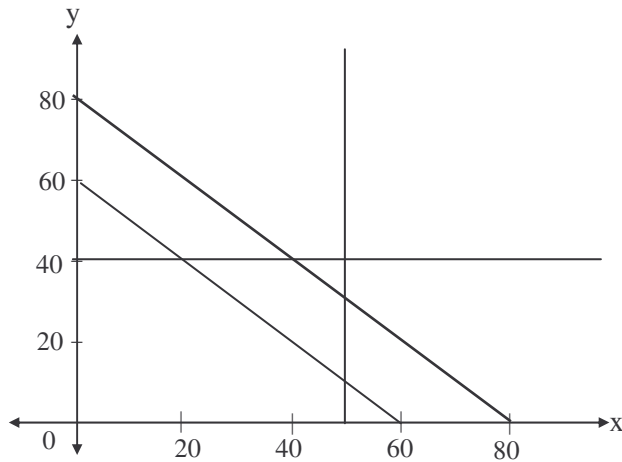
$$1.3 \quad P = 800x + 500y \Leftrightarrow y = -\frac{800}{500}x + \frac{P}{500}$$



Draw a search line with a gradient of $-\frac{800}{500}$. The highest point in the feasible region through which the search line passes is the point $(60;20)$.

The maximum profit is $P = 800(60) + 500(20)$
 $= 48000 + 10000$
 $= 58000$ Rands

2.



A factory makes two types of beds, type A and type B. Each month x of type A and y of type B are produced.

The following constraints control monthly production:

- (i) Not more than 50 beds of type A and 40 beds of type B can be made.
- (ii) To show a profit at least 60 beds in all must be made.
- (iii) The maximum number of beds that can be produced is 80.

- 2.1 The diagram shows the four constraints. Write down in terms of x and y the inequalities that represent these constraints .
- 2.2 Copy the given diagram into your answer book and shade the feasible region.

2.3 If the objective function is given by the equation $y = -2x + \frac{P}{150}$, where P is the monthly profit in rands, what is the profit per bed of the two types of bed..

2.4 How many of each type of bed must be produced per month to maximise profit?
 What is the maximum profit?

2.5 Explain how the production would be affected if the objective function was $y = -x + \frac{P}{150}$.

Solution :

2.1 $0 \leq x \leq 50$, $0 \leq y \leq 40$
 $x + y \geq 60$, $x + y \leq 80$

2.2

2.3 $P = 300x + 150y$

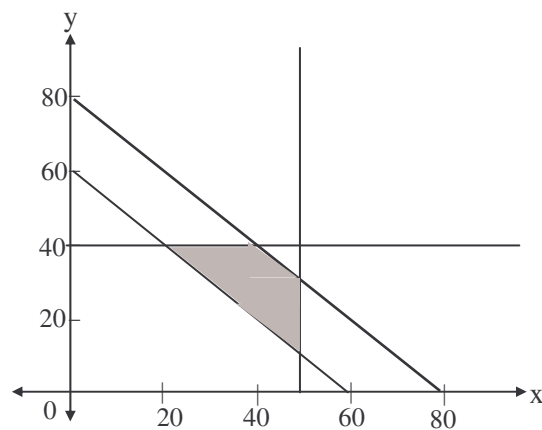
Profit on type A is R300
 and on type B is R150

2.4 Draw a search line with gradient -2 .
 The highest point in the feasible region through which the search line passes is $(50:30)$.

\therefore 50 of type A and 30 of type B must be produced to maximise profit.

The maximum profit is $P = 300(50) + 150(30) = 19500$ Rands

2.5 Since the gradient of this objective function is -1 , the upper limit of the search line will coincide with the line $x + y = 80$. In this case all the ordered pairs (x and y being whole numbers) satisfying the line within the feasible region will give the same maximum profit.



Practice Work

- Sketch the following constraints by clearly indicating all intercepts with the axes and the feasible region :

$$2 \leq x \leq 6 \quad ; \quad y \geq 1 \quad ; \quad 3x + 2y \geq 12 \quad ; \quad 9y + 7x \leq 63$$

Use the objective function $P = 3x + 2y$ to maximise P with respect to the feasible region.

- A furniture manufacturer produces two types of display cabinets, type X and type Y. On a weekly basis he must produce at least 2 of each type, but not more than 5 of type X or more than 6 of type Y. It takes 4 hours to produce type X and 5 hours for type Y in a 40 hour working week. At least 12 workers are needed with 2 working on type X and 3 on type Y at any one time.

- Represent the above information as a system of inequalities .
- Draw a graph of the system and indicate the feasible region clearly.
- If the profit (P) on type X is R800 and on type Y is R1000, write down the objective function in the form $P = ax + by$.
- Determine the number of each type that must be produced each week to make a maximum profit. Determine the maximum profit.

- In the accompanying sketch there is a set of inequalities that leads to the feasible region PQRS as shown by the shaded area. Use the graph to answer the following questions :

- Write down the set of inequalities that is represented by the feasible region .
- Maximise $3x + 2y$ for the given feasible region.
- The co-ordinates of point R minimise the function value of k in $y = mx + k$.
Write down the possible values of m.

